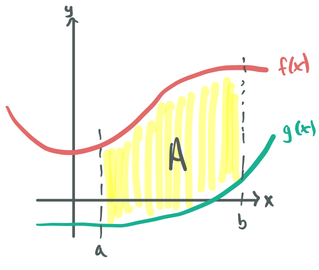
Area between Curves

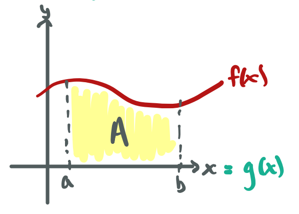
if f, g are continuous on $[a,b]$ with $f(x) \geq g(x)$, then the area A b/w curves $f(x)$ and $g(x)$ is denoted $\longrightarrow$

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

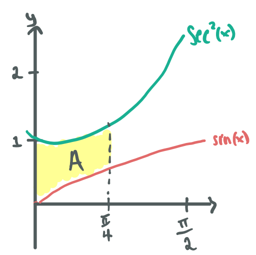


... But this isn't really new is it?
We've been doing this the whole time just with $g(x)$ being the x-axis: $g(x) = 0$

$$A = \int_a^b \left[f(x) - \cancel{g(x)}^{0} \right] dx$$



ex Find the area of the region A between $y = \sec^2(x)$ and $y = \sin(x)$ from $x = 0$ to $x = \frac{\pi}{4}$ & bounded by y-axis.



$$\int_0^{\frac{\pi}{4}} \left[\sec^2(x) - \sin(x) \right] dx$$

$$= \Big[\tan(x) + \cos(x) \Big]_0^{\frac{\pi}{4}}$$

$$= \left(\tan\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right) \right) - \left(\tan(0) + \cos(0) \right)$$

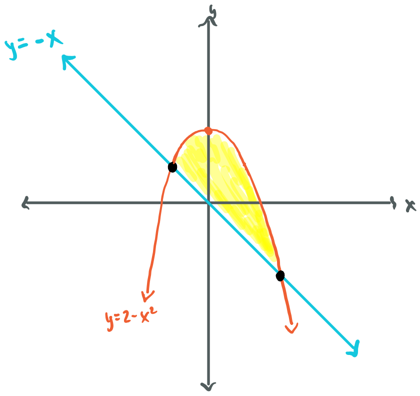
$$= \left(1 + \frac{\sqrt{2}}{2} \right) - \left(0 + 1 \right)$$

$$\boxed{A = \frac{\sqrt{2}}{2}}$$ units squared

Enclosed Areas

what if you're not given bounds a,b?

ex] Find the area enclosed by the Parabola $f(x) = 2 - x^2$ and $g(x) = -x$



Step 1: Find bounds: where $f(x)$ and $g(x)$ intersect

$$2 - x^2 = -x$$
$$\Rightarrow -x^2 + x + 2 = 0$$
$$\Rightarrow x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0 \Rightarrow x = 2,\ x = -1$$

Step 2: Set up integral with bounds

$$\int_{-1}^{2} [f(x) - g(x)]\,dx$$ *where $f(x) \geq g(x)$

Step 3: Plug in associated graphs

$$\int_{-1}^{2} [2 - x^2 - (-x)]\,dx = \int_{-1}^{2} [2 - x^2 + x]\,dx$$

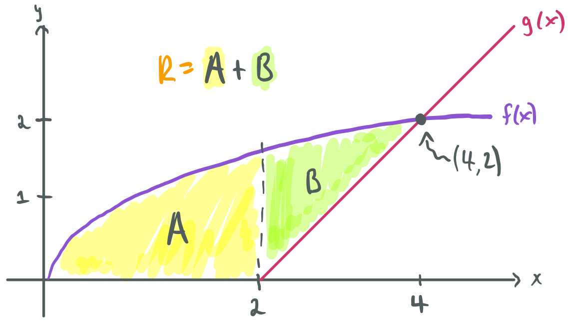
Step 4: Integrate of course!

$$\int_{-1}^{2} [2 - x^2 + x]\,dx = \left[-\frac{x^3}{3} + 2x + \frac{x^2}{2} \right]_{-1}^{2} = \boxed{\frac{9}{2}}$$ units squared

Subregions

We can use subregions to our advantage if a desired area is partitioned accordingly

ex| Find the area of the region R that is bounded above by $f(x)=\sqrt{x}$ and $g(x)=x-2$ and x-axis.



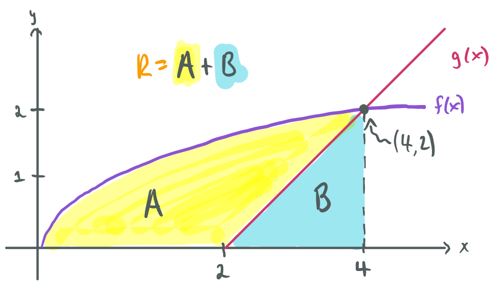
(a) $\sqrt{x} = 2-x$ Find intersect points

$x=4$

(b) $A+B = \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x} - x + 2)\,dx$ OR $\int_0^4 \sqrt{x}\,dx - \int_2^4 x-2\,dx$

(c) $\int_0^2 \sqrt{x}\,dx + \int_2^4 \sqrt{x} - x + 2\,dx \simeq \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_2^4 = \boxed{R = \frac{10}{3}}$ units squared

Using Geometry

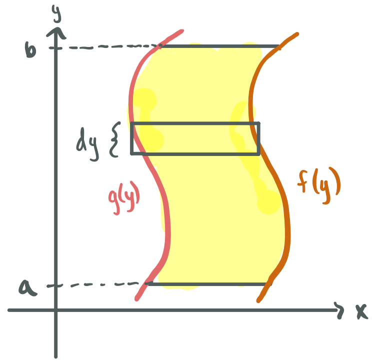


B is a triangle of Area $= \frac{1}{2}(2)(2) = 2$

$R = \int_0^4 f(x)\,dx - \frac{1}{2}(2)(2) = \int_0^4 \left[\sqrt{x}\right] dx - 2 = \left[\frac{2}{3}x^{3/2}\right]_0^4 - 2 = \boxed{R = \frac{10}{3}}$

Integrating with Respect to y

All of our Rectangles have been vertically oriented with **dx** (the width of each Rectangle) becoming infinitesimally small (**skinny**). But we can also integrate using horizontally oriented Rectangles with **dy** (the height of each Rectangle) becoming infinitesimally small (**short**).



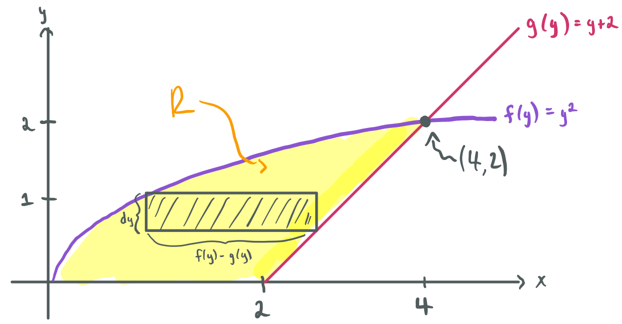
$$A = \int_a^b \left[f(y) - g(y)\right] dy$$

* with $f(y) \geq g(y)$ within $[a, b]$

$$A = \int_a^b \begin{pmatrix}\text{upper}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{lower}\\ \text{function}\end{pmatrix} dx, \qquad a \leq x \leq b$$

$$A = \int_c^d \begin{pmatrix}\text{right}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{left}\\ \text{function}\end{pmatrix} dy, \qquad c \leq y \leq d$$

ex| Find the area of the region R that is bounded above by $f(x)=\sqrt{x}$ and $g(x)=x-2$ and x-axis.



ⓐ Rewrite $f(x), g(x)$ in terms of y

$f(x) = y = \sqrt{x} \Rightarrow y^2 = x = f(y)$ ✓

$g(x) = y = x-2 \Rightarrow y+2 = x = g(y)$ ✓

ⓑ Find intersect points $y^2 = y+2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y=2, \cancel{y=-1} \Rightarrow y = 2$ ✓

note: this step isn't needed

$\Rightarrow 2^2 = x = 4 \Rightarrow (4,2)$

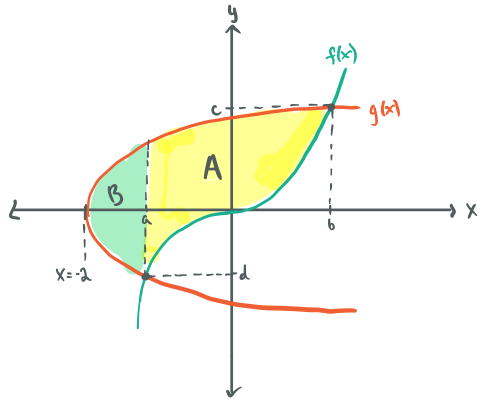
ⓒ Set up integral & ⓓ Integrate

$$\int_0^2 \left[y+2 - y^2\right] dy = \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right]_0^2 = \left(\frac{1}{2}(2)^2 + 2(2) - \frac{1}{3}(2)^3\right) - \cancel{\left(\frac{1}{2}(0)^2 + 2(0) - \frac{1}{3}(0)^3\right)}^{\,0}$$

$$= 2 + 4 - \frac{8}{3} = \boxed{R = \frac{10}{3}}$$ ✓

☆ Note this is the same answer as the previous when we used subregions with respect to dx. Integrating with respect to y only took one integration whereas Integrating with respect to x took two separate integrations.

ex| Find the area of the region R that is bounded above by $f(x) = x^3$ and $g(y) = y^2 - 2$



$R = A + B$

With respect to x:

1. Seperate region R into two subregions A, B
2. Write functions in terms of x
   $y = x^3$
   $x = y^2 - 2 \Rightarrow g(x) = \begin{cases} \sqrt{x+2}, & y \geq 0 \\ -\sqrt{x+2}, & y < 0 \end{cases}$
3. Find intersections a, b
4. Set up integral & Integrate

$$\boxed{2\int_{-2}^{a} g(x)\,dx + \int_{a}^{b} \left[g(x) - f(x)\right] dx}$$

With respect to y:

1. Write functions in terms of y
   $g(y) = x = y^2 - 2 \Rightarrow g(y) = y^2 - 2$
   $f(y) = x^3 = y \Rightarrow f(y) = y^{1/3}$
2. Find intersections c, d
3. Set up integral & Integrate

$$\boxed{\int_{c}^{d} \left[f(y) - g(y)\right] dy}$$

Comparison

$x = y^2 - 2$

$y_1 = \pm\sqrt{x+2}$ and $y_2 = x^3$

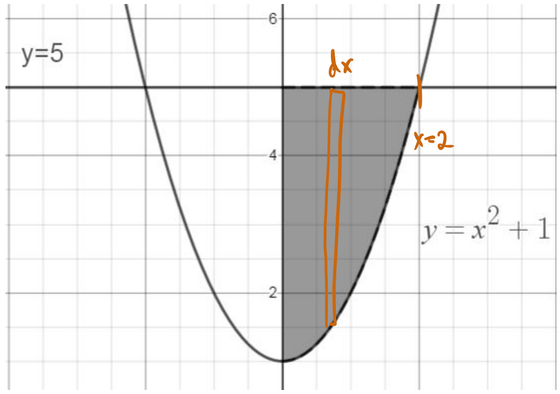
$y_1 = y_2$ at $y = 1.79$ and $y = -1$ (calculator)

$x = 1.21$ and $x = -1$

x: $\int_{-1}^{1.21} \left(\sqrt{x+2} - x^3\right) dx + 2\int_{-2}^{-2} \sqrt{x+2}\, dx = 4.21$

y: $\int_{-1}^{1.79} \sqrt[3]{y} - (y^2-2)\, dy = \int_{-1}^{1.79} \sqrt[3]{y} + y^2 + 2\, dy = 4.21$ ~~A~~ way Easier

AP Calculus BC **Assignment: 7.1-7.2**



$\int_0^2 5-(x^2+1)\,dx$

$= \int_0^2 4-x^2\,dx = \Big[4x-\frac{1}{3}x^3\Big]_0^2$

1. For the figure above, the area of the shaded region is

$= 8-\frac{8}{3} = \frac{24}{3}-\frac{8}{3} = \frac{16}{3}$

(a) $\frac{14}{3}$ (b) $\frac{16}{3}$ (c) $\frac{28}{3}$ (d) $\frac{32}{3}$ (e) $\frac{65}{3}$

$\int_a^b g(x)+5-g(x)\,dx$

$= \Big[5x\Big]_a^b$

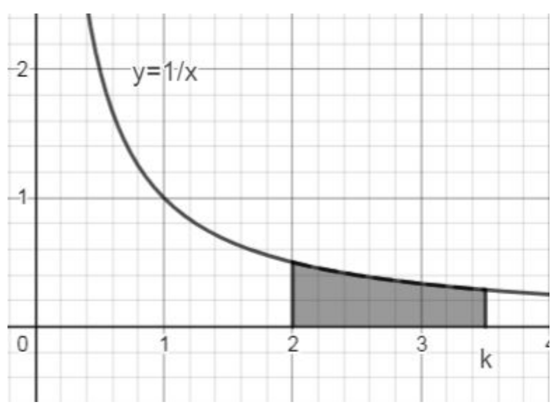
2. If, for all real number x, $f(x) = g(x) + 5$, then on any interval $[a, b]$ the area of the region between the graphs of f and g is

(a) 5 (b) $5a + 5b$ (c) $5b - 5a$ (d) $5a - 5b$ (e) $5ab$

3. The area of the region enclosed by the graphs of $y = e^{x^2} - 2$ and $y = \sqrt{4 - x^2}$ is

$2\cdot\left(\int_0^{1.137}\sqrt{4-x^2}\,dx - \int_{0.833}^{1.137} e^{x^2}-2\,dx + \int_{0.833}^{0} e^{x^2}-2\,dx\right)$

(a) 2.525 (b) 4.049 (c) 4.328 (d) 5.050 (e) 6.289



$\int_2^k \frac{1}{x}\,dx = \Big[\ln(x)\Big]_2^k = \ln(k)-\ln(2) = \ln(4)$

$\ln\left(\frac{k}{2}\right) = \ln(4) \Rightarrow \frac{k}{2} = 4$

4. For the figure above, the area of the shaded region is $ln4$ when k is $\Rightarrow k = 8$

(a) 4 (b) 8 (c) e (d) e^2 (e) e^3

$y'(x) = -e^{2-x}$

$y'(1) = -e^{1}$

$y - y_1 = m(x - x_1)$

$y - e = -e(x-1)$

$y = -e(x-1) + e$

5. The tangent line to the graph $y = e^{2-x}$ at the point $(\overset{x_1}{1}, \overset{y_1}{e})$ intersects both coordinate axes. What is the area of the triangle formed by this tangent line and the coordinate axes?

$\frac{1}{2}\underbrace{(2)}_{b}\underbrace{(2e)}_{h} = 2e$

(a) $2e$ (b) $e^2 - 1$ (c) e^2 (d) $2e\sqrt{e}$ (e) $4e$

6. The area of the region between the graph of $y = 3x^2 + 2x$ and the x-axis from $x = 1$ to $x = 3$ is

$\int_1^3 3x^2 + 2x \, dx = \left[x^3 + x^2\right]_1^3 = (27+9) - (2) = 34$

(a) 36 (b) 34 (c) 31 (d) 26 (e) 12

7. The region bounded by the x-axis an the part of the graph of $y = cos\,(x)$ and $x = \frac{\pi}{2}$ is divided into two regions by the line $x = c$. If the area of the region for $0 \le x \le c$ is equal to the area of the region for $c \le c \le \frac{\pi}{2}$, then c must be

$\int_0^{\frac{\pi}{2}} \cos(x)\, dx = 1$ $\int_0^{c} \cos(x)\, dx = \frac{1}{2} \Rightarrow \sin(c) = \frac{1}{2} \Rightarrow c = \arcsin(\frac{1}{2}) =$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{9}$ (e) $\frac{5\pi}{18}$

8. Let R be the region in the fourth quadrant enclosed by the x-axis and the curve $y = x^2 - 2kx$, where $k > 0$. If the area of the region R is 36, then the value of k is

$x^2 - 2kx = 0$ $x = 0$ $x = 2k$

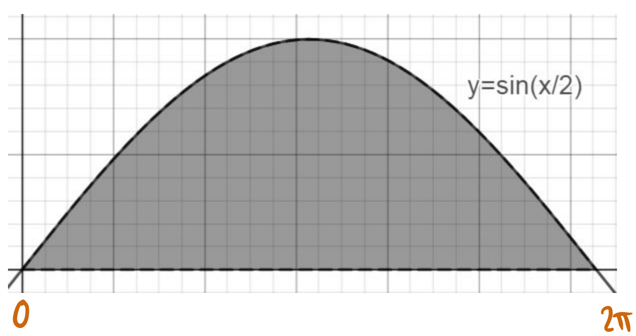
$\int_0^{2k} x^2 - 2kx \, dx = 36$

(a) 2 (b) 3 (c) 4 (d) 6 (e) 9

$\left[\frac{1}{3}x^3 - kx^2\right]_0^{2k} = \left(\frac{8}{3}k^3 - 4k^3\right)$

$\frac{8}{3}k^3 - \frac{12}{3}k^3 = -\frac{4}{3}k^3 = 36$

$k = \sqrt[3]{-\frac{3}{4}\cdot 36} = +3$



9. What is the area of the shaded region in the figure above?

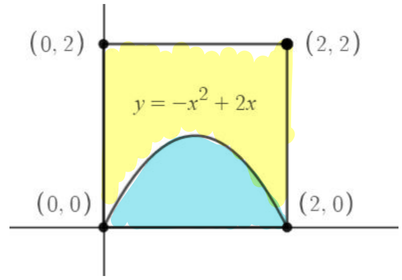
(a) 2 (b) π (c) 4 (d) $2\pi - 1$ (e) 2π

$\int_0^{2\pi} \sin(\frac{x}{2})\, dx = \left[-2\cos(\frac{x}{2})\right]_0^{2\pi}$

$= (2) - (-2) = 4$

$4 - \frac{4}{3} = \frac{8}{3}$

$\int_0^2 -x^2 + 2x\, dx = \left[-\frac{1}{3}x^3 + x^2\right]_0^2 = \frac{4}{3}$

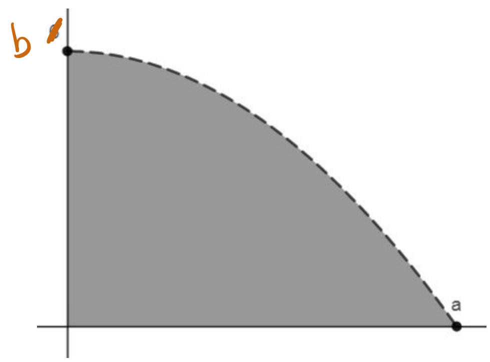


$P(E) = \frac{(8/3)}{4} = \frac{2}{3}$

$A_{square} = 4$

10. As shown in the figure above, a square with vertices $(0,0)$, $(2,0)$, $(2,2)$, and $(2,0)$ is divided into two regions by the graph $y = -x^2 + 2x$. If a point is picked at random from inside the square, what is the probability that the point lies in the region above the parabola?

(a) 0 (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$ (e) $\frac{2}{3}$



11. Let $f(x)$ be a continuous function and let A be the area of the shaded region in the figure above. Which of the following must be true?

I. $A = \int_0^a f(x)\, dx$ ✓

II. $A = \int_0^b f^{-1}(x)\, dx$ ✓

III. $A = \int_0^b f^{-1}(y)\, dy$ ✗

(a) ~~I only~~ (b) II only (c) III only
(d) I and II only (e) I, II, and III

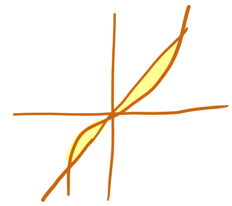
$\int_{1.17}^{3} \arccos(\frac{y}{3})\,dy + \int_{0}^{1.17} y\,dy = 1.39 + 0.68$

12. The area of the region in the first quadrant enclosed by the y-axis and the graphs of $y = 3cosx$ and $y = x$ is $3\cos(x) = x$ @ $x = 1.17$
 $x = \arccos(\frac{y}{3})$

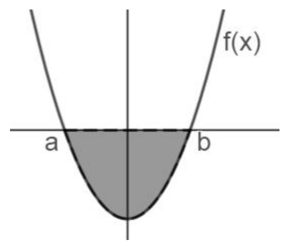
 (a) 1.170 (b) 1.571 (c) 2.078 (d) 3.142 (e) 3.447

13. The total area between the curves $y = x^3$ and $y = x$ is

 (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) $\frac{3}{2}$ (e) 4



$2\cdot\int_0^1 x - x^3\,dx = \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{2}$



14. If f is continuous function shown in the figure above, then the are aof the shaded region is

 (a) $\int_a^b f(x)\,dx$ (b) $\int_b^a f(x)\,dx$ (c) $\int_b^{-a} f(x)\,dx$

 (d) $\int_{-a}^b f(x)\,dx$ (e) $\int_{-b}^{-a} f(x)\,dx$